THE BICOVARIANT DIFFERENTIAL CALCULUS ON THE κ-POINCARÉ GROUP AND ON THE κ-MINKOWSKI SPACE

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Abstract. The bicovariant differential calculus on the four-dimensional κ-Poincaré group and the corresponding Lie-algebra like structure are described. The differential calculus on the n-dimensional κ-Minkowski space covariant under the action of the κ-Poincaré group is constructed.

I. Introduction

In this note we briefly sketch the construction of the differential calculus on the κ-Poincaré group. We obtain the corresponding Lie algebra structure and prove its equivalence to the κ-Poincaré algebra. We sketch the construction of the differential calculus on the n-dimensional κ-Minkowski space covariant under the action of the κ-Poincaré group. Full proofs and discussions of the further properties of these differential calculi will be published elsewhere.

II. The bicovariant calculus on the κ-Poincaré group

The κ-Poincaré group $\mathcal{P}_\kappa$ is the Hopf *-algebra defined as follows [1]. Consider the universal *-algebra with unity generated by the self-adjoint elements $\Lambda^\mu{}_\nu$, x^μ subject to the following relations:

$$\begin{aligned} &[x^\mu, x^\nu] = \frac{i}{\kappa}(\delta_0{}^\mu x^\nu - \delta_0{}^\nu x^\mu), \\ &[\Lambda^\mu{}_\nu, x^\rho] = -\frac{i}{\kappa}((\Lambda^\mu{}_0 - \delta_0{}^\mu)\Lambda^\rho{}_\nu + (\Lambda^0{}_\nu - \delta_\nu{}^0)g^{\mu\rho}) \end{aligned} \tag{1}$$

* Supported by KBN grant 2 P 302 217 06 p 02
** Supported by KBN grant 2 P 302 867 06

here $g_{\mu\rho} = g^{\mu\rho} = \mathrm{diag}\,(+,-,-,-)$ is the metric tensor.

The comultiplication, antipode and counit are introduced as follows:

$$\begin{aligned}
&\varDelta(\varLambda^\mu{}_\nu) = \varLambda^\mu{}_\rho \otimes \varLambda^\rho\nu,\\
&\varDelta(x^\mu) = \varLambda^\mu{}_\nu \otimes v^\nu + x^\mu \otimes I,\\
&S(\varLambda^\mu{}_\nu) = \varLambda_\nu{}^\mu; \qquad S(x^\mu) = -\varLambda_\nu{}^\mu x^\nu,\\
&\varepsilon(\varLambda^\mu{}_\nu) = \delta^\mu{}_\nu; \qquad \varepsilon(x^\mu) = 0.
\end{aligned} \tag{2}$$

The starting point in our construction of the bicovariant *-calculi on the four dimensional κ-Poincaré group is the Woronowicz theory of differential calculi on quantum groups [2]. The main ingredient of this approach is the choice of a right ideal in $\ker\varepsilon$, which is invariant under the adjoint action of the group. The adjoint action is defined as follows:

$$ad(a) = \sum_k b_k \otimes S(a_k)c_k \tag{3}$$

here

$$(\varDelta \otimes I) \circ \varDelta(a) = (I \otimes \varDelta) \circ \varDelta(a) = \sum_k a_k \otimes b_k \otimes c_k.$$

In the classical case, the ideal under consideration is $(\ker\varepsilon)^2$. In order to obtain as slight a deformation of the classical calculus as possible, we start with the generators of $(\ker\varepsilon)^2$. However, it appears that they do not form a multiplet under the adjoint action of the κ-Poincaré group. To cure this, we modify them by adding the appropriate κ-dependent terms. Due to noncommutativity, the ideal generated in this way is identical with the whole $\ker\varepsilon$. Therefore it appears necessary to subtract from the set spanned by the new generators some ad-invariant terms (as a consequence the resulting calculus contains more invariant forms than its classical counterpart). Finally, we arrive at the following

Theorem 1. *Let $\mathcal{R} \subset \ker\varepsilon$ be the right ideal generated by the following elements:*

$$\begin{aligned}
&(\varLambda^\alpha{}_\beta - \delta^\alpha{}_\beta)(\varLambda^\mu{}_\nu - \delta^\mu{}_\nu); \quad \widetilde{\varDelta}^{\mu\nu\alpha} \equiv \varDelta^{\mu\nu\alpha} - \frac{1}{6}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\beta\rho\sigma\delta}\varDelta^{\rho\sigma\delta},\\
&\tilde{x}^{\mu\nu} = x^{\mu\nu} - \frac{1}{4}g^{\mu\nu}x^\alpha{}_\alpha
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
&\varDelta^{\mu\nu\alpha} = x^\alpha(\varLambda^\mu{}_\nu - \delta^\mu{}_\nu) - \frac{i}{\kappa}[\delta^0{}_\nu(\varLambda^{\mu\alpha} - g^{\mu\nu}) + \delta^\mu{}_0(\varLambda^\alpha{}_\nu - \delta^\alpha{}_\nu)],\\
&x^{\mu\nu} \equiv x^\mu x^\nu + \frac{i}{\kappa}(g^{\mu\alpha}x^0 - g^{0\mu}x^\nu).
\end{aligned}$$

Then $\mathcal{R}$ has the following properties:

(i) *$\mathcal{R}$ is ad-invariant, $ad(\mathcal{R}) \subset \mathcal{R} \otimes \mathcal{P}_\kappa$,*
(ii) *for any $a \in \mathcal{R}$, $S(a)^* \in \mathcal{R}$,*
(iii) *$\ker\varepsilon/\mathcal{R}$ is spanned by the following elements:*

$$\begin{aligned}
&x^\mu; \qquad \varLambda^\mu{}_\nu - \delta^\mu{}_\nu, \quad \mu < \nu; \qquad \varphi \equiv x^\alpha{}_\alpha = x^2 + \frac{3i}{\kappa}x^0,\\
&\varphi_\mu \equiv \varepsilon_{\mu\nu\alpha\beta}\varDelta^{\nu\alpha\beta}.
\end{aligned} \tag{5}$$

It is easy to conclude from (iii) that our calculus is fifteen-dimensional. Having established the structure of $\mathcal{R}$, we can now follow closely the Woronowicz construction. The basis of the space of the left-invariant 1-forms consists of the following elements:

$$\begin{aligned}
&\omega^\mu{}_\nu \equiv \pi r^{-1}[I \otimes (\Lambda^\mu{}_\nu - \delta^\mu{}_\nu)] = \Lambda_\alpha{}^\mu d\Lambda^\alpha{}_\nu, \\
&\omega^\mu \equiv \pi r^{-1}[I \otimes x^\mu] = \Lambda_\alpha{}^\mu dx^\alpha, \\
&\omega \equiv \pi r^{-1}[I \otimes \varphi] = d\varphi - 2x_\mu dx^\mu, \\
&\Omega_\mu \equiv \pi r^{-1}[I \otimes \varphi_\mu] = \varepsilon_{\mu\nu\alpha\beta}\Lambda_\sigma{}^\nu \omega^\beta \Lambda^{\sigma\alpha} - \frac{2i}{\kappa}\varepsilon_{0\mu\alpha\beta}\omega^{\alpha\beta}.
\end{aligned} \tag{6}$$

The next step is to find the commutation rules between the invariant forms and generators of $\mathcal{P}_\kappa$. The detailed calculations result in the following formulae:

$$\begin{aligned}
&[\Lambda^\alpha{}_\beta, \omega^\mu{}_\nu] = 0, \\
&[x^\alpha, \omega^\mu{}_\nu] = -\frac{i}{\kappa}(\delta^0{}_\nu \Lambda^\alpha{}_\rho \omega^{\mu\rho} + \delta^\mu{}_0 \Lambda^\alpha{}_\rho \omega^\rho{}_\nu - \Lambda^\alpha{}_\nu \omega^\mu{}_0 - \Lambda^{\alpha\mu}\omega^0{}_\nu) \\
&\qquad\qquad - \frac{1}{6}\varepsilon^\mu{}_\nu{}^{\rho\gamma}\Lambda^\alpha{}_\rho \Omega_\gamma, \\
&[\Lambda^\mu{}_\nu, \omega^\alpha] = -\frac{i}{\kappa}(\delta^0{}_\nu \Lambda^\mu{}_\rho \omega^{\rho\alpha} + \Lambda^\mu{}_0 \omega^\alpha{}_\nu) - \frac{1}{6}\varepsilon^\rho{}_\nu{}^{\alpha\gamma}\Lambda^\mu{}_\rho \Omega_\gamma, \\
&[x^\mu, \omega^\alpha] = -\frac{1}{4}\Lambda^{\mu\alpha}\omega + \frac{i}{\kappa}(\Lambda^{\mu\alpha}\omega^0 - \delta_0{}^\alpha \Lambda^\mu{}_\beta \omega^\beta), \\
&[\Lambda^\mu{}_\nu, \omega] = \frac{4}{\kappa^2}\Lambda^\mu{}_\rho \omega^\rho{}_\nu, \\
&[x^\mu, \omega] = \frac{4}{\kappa^2}\Lambda^\mu{}_\rho \omega^\rho, \\
&[\Lambda^\alpha{}_\beta, \Omega_\mu] = 0, \\
&[x^\alpha, \Omega_\mu] = \frac{3}{\kappa^2}\varepsilon_{\mu\beta\rho\tau}\Lambda^{\alpha\beta}\omega^{\rho\tau} - \frac{i}{\kappa}\delta^0{}_\mu \Lambda^{\alpha\beta}\Omega_\beta + \frac{i}{\kappa}\Lambda^\alpha{}_\mu \Omega_0.
\end{aligned} \tag{7}$$

Then, following Woronowicz's paper [2], we can construct the right-invariant forms:

$$\begin{aligned}
&\eta^\mu{}_\nu = \omega^\beta{}_\gamma \Lambda^\mu{}_\beta \Lambda_\nu{}^\gamma, \\
&\eta^\mu = -\omega^\beta{}_\gamma \Lambda_\rho{}^\gamma x^\rho \Lambda^\mu{}_\beta + \omega^\beta \Lambda^\mu{}_\beta, \\
&\eta = \omega, \\
&\Theta_\mu = \Omega_\nu \Lambda_\mu{}^\nu
\end{aligned} \tag{8}$$

This concludes the description of the bimodule Γ of 1-forms on $\mathcal{P}_\kappa$. The external algebra can now be constructed as follows [2]. On $\Gamma^{\otimes 2}$ we define a bimodule homomorphism σ such that

$$\sigma(\omega \otimes_{\mathcal{P}_\kappa} \eta) = \eta \otimes_{\mathcal{P}_\kappa} \omega \tag{9}$$

for any left-invariant $\omega \in \Gamma$ and any right-invariant $\eta \in \Gamma$. Then, by definition,

$$\Gamma^{\wedge 2} = \frac{\Gamma^{\otimes 2}}{\ker(I - \sigma)}. \tag{10}$$

Higher external powers of Γ can be constructed in a similar way [2]. Eqs. (8)–(10) allow us to calculate the external product of left-invariant 1-forms. The results read:

$$
\begin{aligned}
&\omega \wedge \omega = 0,\\
&\omega^{\mu}{}_{\nu} \wedge \omega^{\alpha}{}_{\beta} + \omega^{\alpha}{}_{\beta} \wedge \omega^{\mu}{}_{\nu} = 0,\\
&\Omega_{\alpha} \wedge \omega^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} \wedge \Omega\alpha = 0,\\
&\Omega_{\mu} \wedge \Omega_{\nu} + \Omega_{\nu} \wedge \Omega_{\mu} = 0,\\
&\omega^{\mu}{}_{\nu} \wedge \omega + \omega \wedge \omega^{\mu}{}_{\nu} - \frac{4}{\kappa^2}\omega^{\sigma}{}_{\nu} \wedge \omega_{\sigma}{}^{\mu} = 0,\\
&\omega \wedge \omega^{\mu} + \omega^{\mu} \wedge \omega - \frac{4}{\kappa^2}\omega^{\mu}{}_{\sigma} \wedge \omega^{\sigma} = 0,\\
&\omega^{\mu} \wedge \omega^{\nu} + \omega^{\nu} \wedge \omega^{\mu} + \frac{i}{\kappa}(\delta_0{}^{\nu}\omega^{\mu}{}_{\rho} \wedge \omega^{\rho} + \delta_0{}^{\mu}\omega^{\nu}{}_{\rho} \wedge \omega^{\rho}) = 0,\\
&\omega^{\alpha} \wedge \omega^{\mu\nu} + \omega^{\mu\nu} \wedge \omega^{\alpha} + \frac{i}{\kappa}(\delta_0{}^{\nu}\omega^{\mu\sigma} \wedge \omega_{\sigma}{}^{\alpha} + \delta_0{}^{\mu}\omega^{\sigma\nu} \wedge \omega_{\sigma}{}^{\alpha}\\
&\qquad\qquad + \omega^{\mu 0} \wedge \omega^{\alpha\nu} + \omega^{0\nu} \wedge \omega^{\alpha\mu}) - \frac{1}{6}\varepsilon^{\alpha\mu\nu\beta}X_{\beta} = 0,\\
&\Omega_{\mu} \wedge \omega + \omega \wedge \Omega_{\mu} - \frac{4}{\kappa^2}\Omega_{\sigma} \wedge \omega^{\sigma}{}_{\mu} - \frac{4}{\kappa^2}X_{\mu} = 0,\\
&\omega^{\mu} \wedge \Omega^{\alpha} + \Omega^{\alpha} \wedge \omega^{\mu} + \frac{i}{\kappa}(\delta_0{}^{\alpha}\Omega_{\rho} \wedge \omega^{\rho\mu} + \Omega_0 \wedge \omega^{\mu\alpha})\\
&\qquad + \frac{1}{12}\varepsilon^{\alpha\mu\rho\sigma}\Omega_{\rho} \wedge \Omega_{\sigma} - \frac{3}{2\kappa^2}\varepsilon^{\alpha\sigma\lambda\tau}\omega^{\mu}{}_{\sigma} \wedge \omega_{\lambda\tau} - \frac{1}{4}g^{\mu\alpha}Y = 0.
\end{aligned}
\tag{11}
$$

The following notation has been introduced in the above formulae:

$$
\begin{aligned}
X_{\mu} &\equiv \varepsilon_{\mu\rho\sigma\delta}[\omega^{\rho} \wedge \omega^{\sigma\delta} + \omega^{\sigma\delta} \wedge \omega^{\rho} + \frac{i}{\kappa}(\delta_0{}^{\delta}\omega^{\sigma\lambda} \wedge \omega_{\lambda}{}^{\rho}\\
&\qquad + \delta_0{}^{\rho}\omega^{\lambda\delta} \wedge \omega_{\lambda} + \omega^{\sigma 0} \wedge \omega^{\rho\delta} + \omega^{0\delta} \wedge \omega^{\rho\sigma})],\\
Y &\equiv \Omega^{\rho} \wedge \omega_{\rho} + \omega_{\rho} \wedge \Omega^{\rho} + \frac{i}{\kappa}\Omega_{\rho} \wedge \omega^{\rho 0}.
\end{aligned}
\tag{12}
$$

Eqs. (11) and (12) have serious consequences. The following property of the classical calculus is usually preserved even in the quantum case: given any basis $\{\omega_i\}$ in the space of (say) left-invariant forms, the basis in $\Gamma^{\wedge 2}$ is spanned by $\omega_i \wedge \omega_j$, $i < j$. This is no longer the case here. The basis in $\Gamma^{\wedge 2}$ consists of the following elements:

$$
\begin{aligned}
&\omega^{\alpha\beta} \wedge \omega^{\mu\nu} \quad (\alpha < \beta,\ \ \mu < \nu,\ \ (\alpha\beta) \neq (\mu\nu),\ \ \alpha < \mu); \quad \omega^{\alpha\beta} \wedge \omega^{\mu};\\
&\omega^{\alpha\beta} \wedge \omega; \omega^{\alpha\beta} \wedge \Omega^{\mu}, \quad \omega^{\alpha} \wedge \omega^{\mu} \ \ (\alpha < \mu); \quad \omega^{\alpha} \wedge \omega;\\
&\omega^{\alpha} \wedge \Omega^{\mu}; \quad \omega \wedge \Omega^{\mu}; \quad \Omega^{\alpha} \wedge \Omega^{\mu} \ \ (\alpha < \mu); \quad X_{\mu}; \ \ Y.
\end{aligned}
$$

Thus there are five more elements than it is generically expected.

To complete our exterior calculus, we derive the Cartan–Maurer equations:

$$
\begin{aligned}
&d\omega^{\mu}{}_{\nu} = \omega_{\sigma}{}^{\mu} \wedge \omega^{\sigma}{}_{\nu},\\
&d\omega^{\mu} = \omega_{\sigma}{}^{\mu} \wedge \omega^{\sigma},\\
&d\omega = 0,
\end{aligned}
\tag{13}
$$

In order to obtain the counterpart of the classical Lie algebra, we introduce the left-invariant fields. They are defined by the formula

$$da = \frac{1}{2}(\chi_{\mu\nu} * a)\omega^{\mu\nu} + (\chi_\mu * a)\omega^\mu + (\chi * a)\omega + (\lambda_\mu * a)\Omega^\mu \tag{14}$$

where, for any linear functional φ on $\mathcal{P}_\kappa$,

$$\varphi * a \equiv (I \otimes \varphi)\Delta(a). \tag{15}$$

The product of two functional φ_1, φ_2 is defined by the standard duality relation

$$\varphi_1\varphi_2(a) \equiv (\varphi_1 \otimes \varphi_2)\Delta(a). \tag{16}$$

Finally, we apply the external derivative to both sides of eq. (14). Using the fact that $d^2a = 0$ and nullifying the coefficients in front of basis elements of $\Gamma^{\wedge 2}$, we find the quantum Lie algebra

$$X^\mu: \ \lambda_\mu\Big(1 - \frac{4}{\kappa^2}\chi\Big) = -\frac{1}{12}\varepsilon_\mu{}^{\alpha\rho\sigma}\chi_\alpha\chi_{\rho\sigma}, \tag{17a}$$

$$Y: \ \lambda^\mu\chi_\mu = 0, \tag{17b}$$

$$\omega^{\alpha\beta} \wedge \omega: \ [\chi_{\alpha\beta}, \chi] = 0, \tag{17c}$$

$$\begin{aligned} \omega^{\alpha\beta} \wedge \Omega^\mu: \ [\chi_{\alpha\beta}, \lambda_\mu] = {} & \Big(1 - \frac{4}{\kappa^2}\chi\Big)(g_{\beta\mu}\lambda_\alpha - g_{\alpha\mu}\lambda_\beta) \\ & + \frac{i}{\kappa}\lambda_0(g_{\mu\beta}\chi_\alpha - g_{\mu\alpha}\chi_\beta) \\ & + \frac{i}{\kappa}\delta^0{}_\mu(\lambda_\alpha\chi_\beta - \lambda_\beta\chi_\alpha) \end{aligned} \tag{17d}$$

$$\omega^\alpha \wedge \omega^\mu: \ [\chi_\alpha, \chi_\mu] = 0, \tag{17e}$$

$$\omega^\alpha \wedge \omega: \ [\chi_\alpha, \chi] = 0, \tag{17f}$$

$$\omega^\alpha \wedge \Omega^\mu: \ [\chi_\alpha, \lambda_\mu] = 0, \tag{17g}$$

$$\omega \wedge \Omega^\mu: \ [\lambda, \lambda_\mu] = 0, \tag{17h}$$

$$\Omega^\alpha \wedge \Omega^\mu: \ [\lambda_\alpha, \lambda_\mu] = \frac{1}{6}\varepsilon_{\alpha\mu}{}^{\rho\sigma}\lambda_\rho\chi_\sigma. \tag{17i}$$

In order to simplify the remaining commutation relations we introduce the following notation:

$$\begin{aligned} l_i &= \chi_{i0}, \\ m_i &= \frac{1}{2}\varepsilon_{ijk}\chi_{jk}. \end{aligned} \tag{18}$$

They read:
$\omega^{\alpha\beta} \wedge \omega^\mu$:

$$[m_i, \chi_0] = 0, \tag{19a}$$

$$[m_i, \chi_k] = \Big(1 + \frac{i}{\kappa}\chi_0 - \frac{4}{\kappa^2}\chi\Big)\varepsilon_{ikl}\chi_l, \tag{19b}$$

$$[l_i, \chi_0] = \Big(1 + \frac{i}{\kappa}\chi_0 - \frac{4}{\kappa^2}\chi\Big)\chi_i \tag{19c}$$

$$[l_i,\chi_k]=\Big(1+\frac{i}{\kappa}\chi_0-\frac{4}{\kappa^2}\chi\Big)\delta_{ik}\chi_0, \tag{19d}$$

$\omega^{\alpha\beta}\wedge\omega^{\mu\nu}$:

$$[m_i,m_j]=\Big(1-\frac{4}{\kappa^2}\chi\Big)\varepsilon_{ijk}m_k+\frac{i}{\kappa}(\chi_j l_i-\chi_i l_j)-\frac{6}{\kappa^2}\lambda_0\varepsilon_{ijk}\chi_k, \tag{19e}$$

$$\begin{aligned}[m_i,l_k]&=\Big(1+\frac{i}{\kappa}\chi_0-\frac{4}{\kappa^2}\chi\Big)\varepsilon_{ikj}l_j+\frac{i}{\kappa}\delta_{ik}\chi_j m_j\\&\quad-\frac{i}{\kappa}\chi_k m_i-\frac{3}{\kappa^2}(\lambda_i\chi_k-\delta_{ik}\lambda_0\chi_0-\delta_{ik}\lambda_j\chi_j),\end{aligned} \tag{19f}$$

$$[l_i,l_k]=-\Big(1+\frac{2i}{\kappa}\chi_0-\frac{4}{\kappa^2}\chi\Big)\varepsilon_{ikj}m_j-\frac{6}{\kappa^2}\chi_0\varepsilon_{ikj}\lambda_j. \tag{19g}$$

In the $\kappa\to\infty$ limit, the classical Poincaré algebra for $\chi_{\alpha\beta}$, χ_μ is restored while λ_μ becomes proportional to the Pauli–Lubanski four–vector. It can be checked that χ is in turn proportional to the mass squared Casimir operator. Let us note that the existence of the additional basis elements X^μ, Y of $\Gamma^{\wedge 2}$ results in additional relations (17a) and (17b) expressing (in the $\kappa\to\infty$ limit) the Pauli–Lubanski four–vector in terms of other generators and the orthogonality of Pauli–Lubanski and momentum four–vectors. It is interesting to observe that the analogous relation for the mass squared operator in terms of momenta is not derivable in this way although its validity can be checked. Relations (17d), (17g) and (17i) express (in the $\kappa\to\infty$ limit) the commutation rules for the Pauli–Lubanski four–vector.

Having our calculus constructed, we can now pose the question: what is the relation (if any) between our functionals $\chi_{\mu\nu}$, χ_α and elements of the κ- Poincaré algebra $\widetilde{\mathcal{P}}_\kappa$ [3]? It has been shown recently [4] that $\mathcal{P}_\kappa$ and $\widetilde{\mathcal{P}}_\kappa$ are formally dual. Therefore we expect $\chi_{\alpha\beta}$, χ_μ, λ_μ to be expressible in terms of elements of $\widetilde{\mathcal{P}}_\kappa$. Using, on the one hand, the properties of the left-invariant fields described by Woronowicz and, on the other hand, the duality relations $\mathcal{P}_\kappa\Longleftrightarrow\widetilde{\mathcal{P}}_\kappa$ established in [4], one can prove that the following substitutions reproduce the algebra and coalgebra structure of our quantum Lie algebra:

$$\begin{aligned}
\lambda_0&=\frac{1}{6}P_iM_ie^{\frac{P_0}{\kappa}},\\
\lambda_i&=\frac{1}{6}\Big[\Big(\kappa\,\mathrm{sh}\Big(\frac{P_0}{\kappa}\Big)+\frac{2\kappa}{e}^{\frac{P_0}{\kappa}}\Big)M_i+\varepsilon_{ijk}P_jN_ke^{\frac{P_0}{\kappa}}\Big],\\
m_i&=-ie^{\frac{P_0}{\kappa}}M_i+\frac{6i}{\kappa}\lambda_i,\\
l_i&=-ie^{\frac{P_0}{\kappa}}N_i,\\
\chi_0&=-i\Big(\kappa\,\mathrm{sh}\Big(\frac{P_0}{\kappa}\Big)+\frac{2\kappa}{e}^{\frac{P_0}{\kappa}}\Big),\\
\chi_i&=-iP_ie^{\frac{P_0}{\kappa}},\\
\chi&=-\frac{1}{8}\Big(2\kappa^2\Big(\mathrm{ch}\Big(\frac{P_0}{\kappa}\Big)-1\Big)-\vec{P}^{\,2}e^{\frac{P_0}{\kappa}}\Big),
\end{aligned} \tag{20}$$

where P_μ, M_i, N_i are generators of the κ-Poincaré algebra.

III. The covariant differential calculi on Minkowski spaces

The n-dimensional κ-Poincaré group is defined, [5], by relations (1), (2) where $g_{\mu\nu} = \mathrm{diag}\,(+,-,\ldots,-)$ is an $n \times n$ matrix. Let us introduce the n-dimensional κ-Minkowski space $\mathcal{M}_\kappa$ as the universal $*$-algebra with unity, generated by selfadjoint elements y^μ subject to the following relations:

$$[y^\mu, y^\nu] = \frac{i}{\kappa}({\delta_0}^\mu y^\nu - {\delta_0}^\nu y^\mu). \tag{21}$$

$\mathcal{M}_\kappa$ can be equipped with the structure of the quantum group by defining:

$$\Delta y^\mu = y^\mu \otimes I + I \otimes y^\mu, \qquad S(y^\mu) = -y^\mu; \qquad \varepsilon(y^\mu) = 0. \tag{22}$$

Let us define the bicovariant (with respect to the group structure on $\mathcal{M}_\kappa$) differential calculus on $\mathcal{M}_\kappa$. To this end we choose $\mathcal{R} \subset \ker\varepsilon$ to be the right ideal generated by

$$\tilde{y}^{\mu\nu} \equiv y^\mu y^\nu + \frac{i}{\kappa}(g^{\mu\nu}y^0 - g^{0\mu}y^\nu) - \frac{1}{n}g^{\mu\nu}\Big(y^2 + \frac{(n-1)}{\kappa}iy^0\Big). \tag{23}$$

Then we have

Theorem 2. *1) $\mathcal{R}$ is ad-invariant, $ad(\mathcal{R}) \subset \mathcal{M}_\kappa \otimes \mathcal{R}$; 2) if $a \in \mathcal{R}$, then $S(a)^* \in \mathcal{R}$; 3) $\ker\varepsilon/\mathcal{R}$ is spanned by y^μ and $\varphi \equiv y^2 + \frac{n-1}{\kappa}iy^0$.*

The left-invariant forms are:

$$\begin{aligned} \tau^\mu &= \pi r^{-1}(I \otimes y^\mu) = dy^\mu, \\ \tau &= \pi r^{-1}(I \otimes \varphi) = d\varphi - 2y_\mu dy^\mu. \end{aligned} \tag{24}$$

They appear to be right-invariant, as well. This fact and the commutation rules

$$\begin{aligned} [\tau^\mu, y^\nu] &= \frac{i}{\kappa}g^{0\mu}\tau^\nu - \frac{i}{\kappa}g^{\mu\nu}\tau^0 + \frac{1}{n}g^{\mu\nu}\tau, \\ [\tau, y^\mu] &= -\frac{n}{\kappa^2}\tau^\mu \end{aligned} \tag{25}$$

imply at once the structure of the exterior calculus

$$\tau^\mu \wedge \tau^\nu + \tau^\nu \wedge \tau^\mu = 0, \qquad \tau^\mu \wedge \tau + \tau \wedge \tau^\mu = 0. \tag{26}$$

Let us proceed to the problem whether the calculus obtained is covariant under the action ρ of $\mathcal{P}_\kappa$ on $\mathcal{M}_\kappa$:

$$\begin{aligned} \rho(I) &= I \otimes I, \\ \rho(y^\mu) &= {\Lambda^\mu}_\nu \otimes y^\nu + x^\mu \otimes I \end{aligned} \tag{27}$$

extended by linearity and multiplicativity. Obviously, ρ is a covariant action on $\mathcal{M}_\kappa$, i.e. a homomorphism $\rho : \mathcal{M}_\kappa \to \mathcal{P}_\kappa \otimes \mathcal{M}_\kappa$ satisfying:

$$\begin{aligned} (I \otimes \rho) \circ \rho &= (\Delta \otimes I) \circ \rho, \\ (\varepsilon \otimes I) \circ \rho &= I \end{aligned} \tag{28}$$

In order to define the covariant action of $\mathcal{P}_\kappa$ on the space of differential forms on $\mathcal{M}_\kappa$, we extend ρ to the action $\widetilde{\rho} : \mathcal{M}_\kappa \otimes \mathcal{M}_\kappa \to \mathcal{P}_\kappa \otimes \mathcal{M}_\kappa \otimes \mathcal{M}_\kappa$:

$$\widetilde{\rho}(q) = \sum_{i,j,k} {a_i}^k {b_i}^j \otimes {x_i}^k \otimes {y_i}^j \tag{29}$$

where $q = \sum_i x_i \otimes y_i \in \mathcal{M}_\kappa \otimes \mathcal{M}_\kappa$ and $\rho(x_i) = \sum_k {a_i}^k \otimes {x_i}^k$, $\rho(y_i) = \sum_j {b_i}^j \otimes {y_i}^j$.

Assume now that $\mathcal{N} \in \mathcal{M}_\kappa^2$ is a sub-bimodule such that $\widetilde{\rho}(\mathcal{N}) \subset \mathcal{P}_\kappa \otimes \mathcal{N}$. Then the differential calculus (Γ, d) defined by $\mathcal{N}$ has the following property:

$$\sum_k x_k dy_k = 0 \to \sum_k \rho(x_k)(I \otimes d)\rho(y_k) = 0. \tag{30}$$

Therefore $\widetilde{\rho}_1 : \Gamma \to \mathcal{P}_\kappa \otimes \Gamma$ given by

$$\widetilde{\rho}_1\big(\sum_k x_k dy_k\big) = \sum_k \rho(x_k)(I \otimes d)\rho(y_k) \tag{31}$$

is a well-defined linear mapping from Γ into $\mathcal{P}_\kappa \otimes \Gamma$.

It then follows that, in order to check whether a calculus on $\mathcal{M}_\kappa$ is consistent with the action of $\mathcal{P}_\kappa$ on $\mathcal{M}_\kappa$, it is sufficient to check the property $\widetilde{\rho}(\mathcal{N}) \subset \mathcal{P}_\kappa \otimes \mathcal{N}$.

One can prove that our first order calculus is covariant. In a similar way we can prove that the higher order calculus is also covariant. We conclude that our bicovariant calculus on $\mathcal{M}_\kappa$, defined by eqs. (24)–(26), is covariant under the action of $\mathcal{P}_\kappa$ which reads (cf. eq. (31)) as follows:

$$\begin{aligned} \widetilde{\rho}_1(\tau^\mu) &= {\Lambda^\mu}_\nu \otimes \tau^\nu, \\ \widetilde{\rho}_1(\tau) &= I \otimes \tau. \end{aligned} \tag{32}$$

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